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THEORETICAL INVESTIGATION OF BIREFRINGENCE-DEFORMATION RELATIONS IN PHOTO-ELASTO-PLASTICITY

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Abstract—The general birefringent relations for elasto-plastic deformation are established theoretically. A proposition is advanced which assumes that the dielectric constants of a polymer solid depend upon elastic and plastic strain. From the coincidence of principal axes of the index ellipsoid and the pseudo-strain quadric, the directions of polarization of a given wave-vector are determined. The velocities of the polarized wave and the birefringent effects in photo-elasto-plasticity are specified by the secondary principal pseudo-strains. Within the elastic limit, the general formula obtained is reduced to the well-known formula of photo-elasticity. In the case when the principal axes of elastic and plastic strain are identical, our general formula is reduced to the one proposed by Filon and Jessop [1], Bayoumi and Frankl [3], and Fujii and Tokuoka [4].

1. INTRODUCTION

THERE have been reported several theoretical investigations concerning the birefringent properties of polymer solids beyond the elastic limit. Filon and Jessop [1] showed that in xylonite and celluloid the relative retardation per unit thickness is expressed as a linear combination of stress and strain, and this formula was verified theoretically by Coker and Filon [2] for a two-phase material. Bayoumi and Frankl [3], and Fujii and Tokuoka [4] proposed the same type of formula. The former were lead to it by experimental results using the material Catalin 800, and the latter deduced it on the basis of three propositions and verified experimentally its validity for celluloid. But all the above investigations have to be regarded as semi-empirical and especially the directions of the plane of polarization were not properly taken into consideration.

In this paper, a qualitative proposition is introduced from the microscopic point of view and from that general formulas are deduced theoretically; in particular, the directions of plane polarization of a given wave-vector for the elasto-plastic deformation region are determined.

2. PROPOSITION

The birefringent properties of a non-magnetic transparent material depend entirely upon its dielectric constants for an observing wave-frequency [5, 6].

When an optically isotropic polymer solid, whose temperature is less than its second-order transition point, is deformed, intermolecular changes of angle and distance and the orientations of the molecular chain arise. These variations have their proper dielectric characteristics and produce an optical anisotropy of the material. The dielectric constants may be considered as the macroscopic mean effects of these variations.

On the other hand, the deformation of a polymer solid is the macroscopic result of the accumulation of these geometrical changes of the molecules. Then, as Neumann [7]

proposed, the dielectric constants may be assumed as functions of the deformation state of the material, i.e. elastic and plastic strain.

Now we suppose that the deviation of the dielectric constants from its undeformed natural state is relatively small and the dielectric cross-effect of elastic and plastic strain is negligible in the first approximation. Then elastic and plastic strain may be the independent variables for the dielectric constants.

From the above considerations, we deduce the following conclusion.

Proposition: The dielectric constants of a polymer solid depend upon elastic and plastic strain of the material.

3. BIREFRINGENCE–DEFORMATION RELATIONS

When a polymer solid is deformed, the polymer becomes an anisotropic dielectric for an observing wave-frequency and the electric induction $D_i (i = 1, 2, 3)$ may be expressed linearly by the electric field $E_i (i = 1, 2, 3)$ as

$$D_i = \varepsilon_{ij} E_j \qquad (i = 1, 2, 3), \tag{1}$$

where the coefficients $\varepsilon_{ij} (i, j = 1, 2, 3)$ are the dielectric constants of the material and in the above and following expressions repeated indices indicate summation over the range (1, 2, 3).

We now take the inverse transformation of (1) and write this solution as

$$E_i = \eta_{ij} D_j \qquad (i = 1, 2, 3), \tag{2}$$

where η_{ij} is an (i, j) element of the inverse matrix of (ε_{ij}).

The symmetry conditions

$$\varepsilon_{ij} = \varepsilon_{ji} \qquad \text{and} \qquad \eta_{ij} = \eta_{ji} \qquad (i, j = 1, 2, 3) \tag{3}$$

follow from the requirement that the work done per unit volume in creating a field must be a total differential.

The dielectric constant and the magnetic permeability of vacuum are both unity in the system of Gaussian units and, as for all transparent materials, the magnetic permeability of the given material is assumed to be unity for all states of deformation.

From our proposition introduced in Section 2, the dielectric constants ε_{ij} and thus also the coefficients η_{ij} are functions of the elastic strains e^E_{ij} and the plastic strains e^P_{ij} such that

$$\eta_{ij} = \eta_0 \delta_{ij} + \eta^E_{ij}(e^E_{kl}) + \eta^P_{ij}(e^P_{kl}), \qquad (i, j = 1, 2, 3) \tag{4}$$

in the first approximation, where η_0 is the reciprocal of the dielectric constant ε_0 of the undeformed natural isotropic state of the material and

$$\eta_0 = (v_0/c)^2 = (n_0)^{-2}, \tag{5}$$

where v_0 and c are the ray velocity penetrating into the undeformed material and the velocity in vacuum, respectively, and n_0 is the index of refraction in this state; The symbol δ_{ij}, the Kronecker delta, is defined to have the value one if i equals j, zero if i differs from j.

The birefringent properties of a dielectric are completely determined by Fresnel's ellipsoid

$$\varepsilon_{ij} x_i x_j = 1 \tag{6}$$

or the index ellipsoid

$$\eta_{ij}x_ix_j = 1. \tag{7}$$

The directions of polarization are parallel to the axes of the diametral cross-section being parallel to the wave-front of the index ellipsoid (7), and the index of refraction for a plane polarized wave is the length of the semi-axis perpendicular to itself [5, 6].

Now we consider the elastic index ellipsoid

$$\eta^E_{ij}x_ix_j = \text{constant} \tag{8}$$

and the elastic strain quadric

$$e^E_{ij}x_ix_j = \text{constant}. \tag{9}$$

The coefficients η^E_{ij} depend only on the elastic strains e^E_{ij}, and thus, according to Neumann [7], it must follow that the two quadrics (8) and (9), from considerations of symmetry, have the same principal axes. Taking these axes as coordinate axes, the equations of these quadrics become

$$\eta^E_i x_i^2 = \text{constant} \tag{10}$$

and

$$e^E_i x_i^2 = \text{constant}, \tag{11}$$

where $e^E_i (i = 1, 2, 3)$ are the principal elastic strains.

Then we must have

$$\left.\begin{aligned} \eta^E_1 &= f^E(e^E_1, e^E_2, e^E_3) \\ &\text{and, by symmetry,} \\ \eta^E_2 &= f^E(e^E_2, e^E_3, e^E_1), \\ \eta^E_3 &= f^E(e^E_3, e^E_1, e^E_2), \end{aligned}\right\} \tag{12}$$

where f^E is some function which vanishes when $e^E_1 = e^E_2 = e^E_3 = 0$ and, owing to the symmetry of the two quadrics (10) and (11) about the principal axes, f^E is necessarily a symmetric function of its second and third variables.

Expanding (12) into a Taylor series and neglecting second and higher order terms of strain, we obtain

$$\eta^E_i = \alpha^E e^E_i + \beta^E e^E, \qquad (i = 1, 2, 3), \tag{13}$$

where

$$\left.\begin{aligned} \alpha^E &\equiv \left(\frac{\partial f^E(\xi, \eta, \zeta)}{\partial \xi}\right)_{0,0,0} - \beta^E, \\ \beta^E &\equiv \left(\frac{\partial f^E(\xi, \eta, \zeta)}{\partial \eta}\right)_{0,0,0} = \left(\frac{\partial f^E(\xi, \eta, \zeta)}{\partial \zeta}\right)_{0,0,0} \end{aligned}\right\} \tag{14}$$

and $e^E \equiv e^E_1 + e^E_2 + e^E_3$ is the first elastic strain invariant.

By means of an orthogonal transformation from the coordinate system of the principal axes of the quadrics to the original coordinate system, we have

$$\eta^E_{ij} = \alpha^E e^E_{ij} + \beta^E e^E \delta_{ij}, \qquad (i, j = 1, 2, 3). \tag{15}$$

In the same way, we have the following relations with respect to the plastic part

$$\eta_{ij}^P = \alpha^P e_{ij}^P + \beta^P e^P \delta_{ij}, \qquad (i,j = 1,2,3), \tag{16}$$

where $e^P \equiv e_1^P + e_2^P + e_3^P$ is the first plastic strain invariant. (In general the two systems of the principal axes of elastic and plastic strain quadric differ from each other, nevertheless, we can obtain the relations (15) and (16) by the transformations to the original coordinate system.)

Combining (15), (16) and (4), we have

$$\eta_{ij} = \eta_0 \delta_{ij} + \alpha^E e_{ij}^* + \beta^E e^{**} \delta_{ij}, \qquad (i,j = 1,2,3), \tag{17}$$

where

$$e_{ij}^* \equiv e_{ij}^E + \alpha e_{ij}^P, \qquad (i,j = 1,2,3) \tag{18}$$

may be called pseudo-strains and

$$e^{**} \equiv e^E + \beta e^P, \tag{19}$$

while

$$\alpha \equiv \frac{\alpha^P}{\alpha^E} \qquad \text{and} \qquad \beta \equiv \frac{\beta^P}{\beta^E} \tag{20}$$

are specified for a given material and the observing wave-frequency.

Relation (17) indicates that the index ellipsoid (7) and the pseudo-strain quadric

$$e_{ij}^* x_i x_j = \pm 1 \tag{21}$$

have the same principal axes, where the sign is chosen so as to make the surface real. Taking these axes as coordinate axes, relation (17) is then reduced to

$$\eta_i = \eta_0 + \alpha^E e_i^* + \beta^E e^{**}, \qquad (i = 1,2,3), \tag{22}$$

where $e_i^* (i = 1,2,3)$ are the principal pseudo-strains.

Let us consider the sections of the index ellipsoid and the pseudo-strain quadric defined by the diametral plane

$$k_i x_i = 0 \tag{23}$$

perpendicular to the wave-vector **k**, i.e. parallel to the wave-front.

If we choose the x_3-axis as the direction of the wave-vector in the quadric equations (7) and (21), generality is not lost. Then the equations of the intersectional curve are expressed as

$$\eta_{11} x_1^2 + 2\eta_{12} x_1 x_2 + \eta_{22} x_2^2 = 1 \tag{24}$$

and

$$e_{11}^* x_1^2 + 2e_{12}^* x_1 x_2 + e_{22}^* x_2^2 = \pm 1. \tag{25}$$

Thus the angles between the principal axes of the intersectional curves of second degree, being designated as the secondary principal axes, and the x_1-axis are specified, respectively, as

$$\tan 2\theta_\eta = \frac{2\eta_{12}}{\eta_{11} - \eta_{22}} \qquad \text{and} \qquad \tan 2\theta_{e^*} = \frac{2e_{12}^*}{e_{11}^* - e_{22}^*}. \tag{26}$$

Considering (26) and (17), we may conclude that the secondary principal axes of the two sections have the same directions.

Rotating the coordinate axes to the same secondary principal axes, which are denoted by x_α ($\alpha = 1, 2$), the curves of second degree are represented by

$$\eta'_1 x'^2_1 + \eta'_2 x'^2_2 = 1 \tag{27}$$

and

$$e'^*_1 x'^2_1 + e'^*_2 x'^2_2 = \pm 1, \tag{28}$$

while the coefficients η'_α and the secondary principal pseudo-strains $e'^*_\alpha (\alpha = 1, 2)$ are equal to

$$\eta'_1, \eta'_2 = \tfrac{1}{2}\{\eta_{11} + \eta_{22} \pm \sqrt{[(\eta_{11} - \eta_{22})^2 + 4\eta^2_{12}]}\} \tag{29}$$

and

$$e'^*_1, e'^*_2 = \tfrac{1}{2}\{e^*_{11} + e^*_{22} \pm \sqrt{[(e^*_{11} - e^*_{22})^2 + 4e^{*2}_{12}]}\}. \tag{30}$$

Substituting relation (17) into (29) and considering (30), we obtain

$$\eta'_\alpha = \eta_0 + \alpha^E e'^*_\alpha + \beta^E e^{**}, \qquad (\alpha = 1, 2). \tag{31}$$

These secondary principal axes are parallel to the directions of polarization at this wave-front.

Let us denote by $v'_\alpha (\alpha = 1, 2)$ the velocities of the wave polarized along the x'_α-axis, such that

$$v'^2_1 = c^2 \eta'_2 \qquad \text{and} \qquad v'^2_2 = c^2 \eta'_1. \tag{32}$$

By means of (31) and (32), we obtain

$$\left.\begin{aligned} v'^2_1 &= v_0^2 + c^2(\alpha^E e'^*_2 + \beta^E e^{**}), \\ v'^2_2 &= v_0^2 + c^2(\alpha^E e'^*_1 + \beta^E e^{**}). \end{aligned}\right\} \tag{33}$$

Taking square roots in (33) and neglecting squares of strain, we have

$$\left.\begin{aligned} v'_1 &= v_0 + p e'^*_2 + q e^{**}, \\ v'_2 &= v_0 + p e'^*_1 + q e^{**}, \end{aligned}\right\} \tag{34}$$

where

$$p \equiv \frac{c^2}{2v_0}\alpha^E \qquad \text{and} \qquad q \equiv \frac{c^2}{2v_0}\beta^E.$$

Then the relative retardation of the wave having the given wave-vector is

$$r = v'_1 - v'_2 = p(e'^*_2 - e'^*_1) \tag{35}$$

and the fringe-order per unit thickness is expressed approximately by

$$N = \omega\left(\frac{1}{v'_1} - \frac{1}{v'_2}\right) = A(e'^*_1 - e'^*_2), \tag{36}$$

where ω is the observing wave-frequency and $A \equiv (\omega/v_0^2)p$ is a pseudo-strain-optical coefficient specified for a given material and given wave-frequency.

From the above developments, we can conclude the following two laws:

(1) The directions of polarization at any given wave-front are parallel to the directions of secondary principal pseudo-strain at that wave-front, that is, to the principal axes of the section of the pseudo-strain quadric defined by a plane parallel to the wave-front.

(2) The relative retardation and the fringe-order per unit thickness of the two oppositely polarized waves which have the same wave-front are proportional to the difference of the principal pseudo-strains in the plane of the wave-front.

4. SPECIAL CASES

4.1. *The case of the deformation state within the elastic limit*

In this case the plastic strains vanish. Then the pseudo-strains e_{ij}^{*} coincide with the total strains e_{ij} and e^{**} with the first invariant of the total strain $e \equiv e_{ii}$. Then the pseudo-strain quadric is reduced to the strain quadric.

The velocities of the polarized wave are thus equal to

$$\left.\begin{aligned} v_1'^2 &= v_0^2 + c^2(\alpha^E e_2' + \beta^E e), \\ v_2'^2 &= v_0^2 + c^2(\alpha^E e_1' + \beta^E e) \end{aligned}\right\} \tag{37}$$

in spite of (34), where the directions of polarization of a given wave coincide with the secondary principal axes of the section of the strain quadric defined by a plane parallel to the wave-front and $e_\alpha'(\alpha = 1, 2)$ are the secondary principal strains at the wave-front.

Further, the relative retardation of the wave and the fringe-order per unit thickness are expressed by

$$r = v_1' - v_2' = p(e_2' - e_1') \tag{38}$$

and

$$N = A(e_1' - e_2') \tag{39}$$

respectively.

Equations (37)–(39) are Neumann's relations [7].

If the given elastic body is a mechanically homogeneous isotropic medium, the stress–strain relations are of the form [8]

$$e_{ij} = \frac{-\lambda\delta_{ij}}{2\mu(3\lambda + 2\mu)}\sigma + \frac{1}{2\mu}\sigma_{ij}, \qquad (i, j = 1, 2, 3), \tag{40}$$

where λ and μ are the Lamé constants; $\sigma_{ij}(i, j = 1, 2, 3)$ are the stress components and $\sigma \equiv \sigma_{ii}$ is the first stress invariant.

In this case, the secondary principal axes of the strain quadric coincide with that of the stress quadric and

$$e_1' - e_2' = \frac{1}{2\mu}(\sigma_1' - \sigma_2'), \tag{41}$$

where $\sigma_\alpha'(\alpha = 1, 2)$ are the secondary principal stresses.

The fringe-order per unit thickness is then proportional to the secondary principal stress difference at the wave-front such that

$$N \equiv C(\sigma'_1 - \sigma'_2), \tag{42}$$

where $C \equiv A/2\mu$.

Equation (42) is the well-known "*stress-optical law*" in photo-elasticity.

Thus our general formula includes the formula (42) as a special case of the deformation state within the elastic limit.

4.2. *The case of the deformation state, in which the principal axes of elastic and plastic strain are identical and the wave-vector is proportional to one of these axes.*

In this case, the three systems of the principal axes of the index ellipsoid, and of the elastic and plastic strain quadric are identical and thus the secondary principal axes of the section of the index ellipsoid coincide with those of elastic and plastic strain quadric.

Therefore the expression for the fringe-order per unit thickness is reduced to

$$N = A(e'^E_1 - e'^E_2) + B(e'^P_1 - e'^P_2), \tag{43}$$

where e'^E_α and $e'^P_\alpha (\alpha = 1, 2)$ are the secondary principal elastic and plastic strains, respectively, and $B \equiv \alpha A$.

If, further, the secondary principal elastic strain difference is assumed to be proportional to the secondary principal stress difference, as for a Prandtl–Reuss body, we have

$$e'^E_1 - e'^E_2 = \frac{1}{2\mu}(\sigma'_1 - \sigma'_2). \tag{44}$$

Then (43) takes on the form

$$N = C_1(\sigma'_1 - \sigma'_2) + C_2(e'_1 - e'_2), \tag{45}$$

where

$$e'_\alpha \equiv e'^E_\alpha + e'^P_\alpha \ (\alpha = 1, 2), \quad \text{and} \quad C_1 \equiv \frac{A - B}{2\mu} \quad \text{and} \quad C_2 \equiv B$$

are the stress-optical and the strain-optical coefficient, respectively.

Equation (45) is precisely the formula proposed by Filon and Jessop, Bayoumi and Frankl, and Fujii and Tokuoka.

Thus we can conclude that the birefringent effect is expressed as a linear combination of stress and strain if the principal axes of elastic and plastic strain are identical and the wave-vector is proportional to one of these axes.

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Zusammenfassung—Es werden die allgemeinen Beziehungen der Doppelbrechung bei der elastoplastischen Formänderung theoretisch gegeben. Ein Ansatz wird vorgeschlagen, wonach sich die Dielektrizitätskonstanten eines polymeren, festen Körpers von den elastischen und plastischen Formänderungen abhängen. Auf Grund des Zusammenfallens der Hauptachsen des Indexellipsoides und der Pseudoverzerrungsfläche, werden die Richtungen des polarisierten Lichtes eines gegebenen Wellenvektors bestimmt. Die Geschwindigkeiten des polarisierten Lichtes, sowie die spannungsoptischen Effekte bei der Photoelastoplastizität sind durch die sekundären Hauptpseudoverzerrungen gegeben. Für den Fall innerhalb der Elastizitätsgrenze, wird die aufgestellte allgemeine Formel auf die bekannte Formel der Photoelastizität zurückgeführt. Der Fall, wo die Hauptachsen der elastischen und plastischen Formänderung identisch sind, führt auf die Formel die von Filon und Jessop [1], Bayoumi und Frankl [3], und Fujii und Tokuoka [4] vorgeschlagen wurde.

Абстракт—Установлены теоретически общие отношения двойного лучепреломления для упруго-пластичной деформации. Сделано предположение, что диэлектрические постоянные твердого полимера зависят от упругой и пластической деформации. Определены направления поляризации данной волны-вектора из совпадений главных осей квадрики индекса и квадрики псевдодеформации. Скорости поляризованной волны и эффект двойного преломления детализируются в фото-упруго-пластичной среде вторичными главными псевдодеформациями. В пределах упругости выведенная общая формула приводится к хорошо известной формуле фото-упругости. В случае, когда главные оси упругой и пластической деформации идентичны, наша общая формула приводится к формуле, предложенной группами: Филон и Джессоп [1], Байуми и Франкл [3], и Фуджи и Токуока [4].